Sea f: $\mathbb{R}^4 \to M_2(\mathbb{R})$ la aplicación dada por

$$f(x, y, z, t) = \begin{pmatrix} z-x & 0 \\ t & 2t \end{pmatrix}$$

a) Calcular dimensión, base, ecuaciones paramétricas e implícitas de Ker(f) e Im(f).
b) Calcular la matriz asociada a f respecto de las bases B = {(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)} de $\mathbb{R}^4$ y B' = $\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ de $M_2(\mathbb{R})$.
c) ¿Es f un isomorfismo? Razonar la respuesta a partir de la matriz obtenida en el apartado anterior.

a) $\operatorname{Ker}(f) = \left\{ (x,y,z,t) \;/\; f(x,y,z,t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

$$\begin{pmatrix} z-x & 0 \\ t & 2t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \left.\begin{array}{r} z-x=0 \\ t=0 \\ \cancel{2t=0} \end{array}\right\} \text{Ec. implícitas de Ker}(f)$$

$$\left.\begin{array}{l} x=\alpha \\ y=\beta \\ z=\alpha \\ t=0 \end{array}\right\} \text{Ec. paramétricas} \rightsquigarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\alpha + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\beta$$

$$B_{\operatorname{Ker}(f)} = \{(1,0,1,0), (0,1,0,0)\}$$

$$\dim(\operatorname{Ker}(f)) = 2$$

$B_C = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ base de $\mathbb{R}^4$

$\Downarrow$

$\{f(1,0,0,0), f(0,1,0,0), f(0,0,1,0), f(0,0,0,1)\}$ sist. generador de Im(f)

$$f(1,0,0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$f(0,1,0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad f(0,0,0,1) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

$$f(0,0,1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Sist. generador de $\mathrm{Im}(f)$

$$\left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\}$$

$$\Rightarrow B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \text{ base de } \mathrm{Im}(f)$$

$\dim(\mathrm{Im}(f)) = 2$

$$\mathrm{rg} \begin{pmatrix} \boxed{1 \quad 0} \\ 0 \quad 0 \\ \boxed{0 \quad 1} \\ 0 \quad 2 \end{pmatrix} = 2$$

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\left.\begin{aligned} x &= \alpha \\ y &= 0 \\ z &= \beta \\ t &= 2\beta \end{aligned}\right\} \text{Ec. paramétricas de } \mathrm{Im}(f) \Rightarrow \left.\begin{aligned} y &= 0 \\ 2z - t &= 0 \end{aligned}\right\} \text{Ec. impl. de } \mathrm{Im}(f)$$

b) $B = \{(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\}$

$$f(1,0,0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}_{B_c} = (x_1, x_2, x_3, x_4)_{B'} = (0,0,-1,0)_{B'}$$

$$f(1,1,0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B' = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$f(1,1,1,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (0,0,0,0)_{B'}$$

$$f(1,1,1,1) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = (x_1, x_2, x_3, x_4)$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = x_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_3 &= 0 \\ x_4 &= 0 \\ x_1+x_2+x_4 &= 1 \\ x_2 &= 2 \end{aligned}\right\} \longrightarrow \begin{aligned} &x_3 = 0 \\ &x_4 = 0 \\ &x_1 = 1-2 = -1 \\ &x_2 = 2 \end{aligned}$$

$$f(1,1,1,1) = (-1,2,0,0)_{B'}$$

$$M_{B,B'}(f) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

c) isomorfismo $\equiv$ aplicación lineal biyectiva

$$\Longleftrightarrow \operatorname{rg}(M_{B,B'}(f)) \overset{??}{=} \underset{4}{\underset{\shortparallel}{\dim(\mathbb{R}^4)}} = \underset{4}{\underset{\shortparallel}{\dim(M_2(\mathbb{R}))}}$$

$$\operatorname{rg}(M_{B,B'}(f)) = 2$$

$$\begin{pmatrix} 0 & \cancel{0} & \cancel{0} & -1 \\ 0 & \cancel{0} & \cancel{0} & 2 \\ -1 & \cancel{-1} & \cancel{0} & 0 \\ 0 & \cancel{0} & \cancel{0} & 0 \end{pmatrix}$$

No isomorfismo

$$\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0$$